4.2 Quantum definitions

Quantum uncertainty relations

De Broglie relation	$p = \dfrac{h}{\lambda}$ $\boldsymbol{p} = \hbar \boldsymbol{k}$	(4.1) (4.2)	$p, \boldsymbol{p}$ particle momentum h Planck constant $\hbar$ $h/(2\pi)$ λ de Broglie wavelength $\boldsymbol{k}$ de Broglie wavevector
Planck–Einstein relation	$E = h\nu = \hbar\omega$	(4.3)	E energy ν frequency ω angular frequency $(=2\pi\nu)$
Dispersion[a]	$(\Delta a)^2 = \langle (a - \langle a \rangle)^2 \rangle$ $= \langle a^2 \rangle - \langle a \rangle^2$	(4.4) (4.5)	a, b observables[b] $\langle \cdot \rangle$ expectation value $(\Delta a)^2$ dispersion of a
General uncertainty relation	$(\Delta a)^2 (\Delta b)^2 \geq \dfrac{1}{4} \langle \mathrm{i}[\hat{a}, \hat{b}] \rangle^2$	(4.6)	$\hat{a}$ operator for observable a $[\cdot,\cdot]$ commutator (see page 26)
Momentum–position uncertainty relation[c]	$\Delta p \, \Delta x \geq \dfrac{\hbar}{2}$	(4.7)	x particle position
Energy–time uncertainty relation	$\Delta E \, \Delta t \geq \dfrac{\hbar}{2}$	(4.8)	t time
Number–phase uncertainty relation	$\Delta n \, \Delta \phi \geq \dfrac{1}{2}$	(4.9)	n number of photons ϕ wave phase

[a]Dispersion in quantum physics corresponds to variance in statistics.
[b]An observable is a directly measurable parameter of a system.
[c]Also known as the "Heisenberg uncertainty relation."

Wavefunctions

Probability density	$\mathrm{pr}(x,t)\,\mathrm{d}x = \lvert \psi(x,t) \rvert^2 \,\mathrm{d}x$	(4.10)	pr probability density ψ wavefunction
Probability density current[a]	$j(x) = \dfrac{\hbar}{2\mathrm{i}m}\left(\psi^* \dfrac{\partial \psi}{\partial x} - \psi \dfrac{\partial \psi^*}{\partial x}\right)$ $\boldsymbol{j} = \dfrac{\hbar}{2\mathrm{i}m}\left[\psi^*(\boldsymbol{r})\nabla\psi(\boldsymbol{r}) - \psi(\boldsymbol{r})\nabla\psi^*(\boldsymbol{r})\right]$ $= \dfrac{1}{m}\Re(\psi^* \hat{\boldsymbol{p}} \psi)$	(4.11) (4.12) (4.13)	$\boldsymbol{j}, j$ probability density current $\hbar$ (Planck constant)/(2π) x position coordinate $\hat{\boldsymbol{p}}$ momentum operator m particle mass $\Re$ real part of t time
Continuity equation	$\nabla \cdot \boldsymbol{j} = -\dfrac{\partial}{\partial t}(\psi\psi^*)$	(4.14)	
Schrödinger equation	$\hat{H}\psi = \mathrm{i}\hbar \dfrac{\partial \psi}{\partial t}$	(4.15)	H Hamiltonian
Particle stationary states[b]	$-\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$	(4.16)	V potential energy E total energy

[a]For particles. In three dimensions, suitable units would be particles $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$.
[b]Time-independent Schrödinger equation for a particle, in one dimension.

Operators

Hermitian conjugate operator	$\int (\hat{a}\phi)^* \psi \, dx = \int \phi^* \hat{a}\psi \, dx$	(4.17)	$\hat{a}$ ψ, ϕ	Hermitian conjugate operator normalisable functions
Position operator	$\hat{x^n} = x^n$	(4.18)	* x, y	complex conjugate position coordinates
Momentum operator	$\hat{p_x^n} = \frac{\hbar^n}{\mathrm{i}^n} \frac{\partial^n}{\partial x^n}$	(4.19)	n p_x	arbitrary integer ≥ 1 momentum coordinate
Kinetic energy operator	$\hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$	(4.20)	T $\hbar$ m	kinetic energy (Planck constant)/(2π) particle mass
Hamiltonian operator	$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$	(4.21)	H V	Hamiltonian potential energy
Angular momentum operators	$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$	(4.22) (4.23)	L_z L	angular momentum along z axis (sim. x and y) total angular momentum
Parity operator	$\hat{P}\psi(\boldsymbol{r}) = \psi(-\boldsymbol{r})$	(4.24)	$\hat{P}$ $\boldsymbol{r}$	parity operator position vector

Expectation value

Expectation value[a]	$\langle a \rangle = \langle \hat{a} \rangle = \int \Psi^* \hat{a} \Psi \, dx$ $= \langle \Psi \vert \hat{a} \vert \Psi \rangle$	(4.25) (4.26)	$\langle a \rangle$ $\hat{a}$ Ψ x	expectation value of a operator for a (spatial) wavefunction (spatial) coordinate
Time dependence	$\frac{d}{dt}\langle \hat{a} \rangle = \frac{\mathrm{i}}{\hbar}\langle [\hat{H}, \hat{a}] \rangle + \left\langle \frac{\partial \hat{a}}{\partial t} \right\rangle$	(4.27)	t $\hbar$	time (Planck constant)/(2π)
Relation to eigenfunctions	if $\hat{a}\psi_n = a_n \psi_n$ and $\Psi = \sum c_n \psi_n$ then $\langle a \rangle = \sum \vert c_n \vert^2 a_n$	(4.28)	ψ_n a_n n c_n	eigenfunctions of $\hat{a}$ eigenvalues dummy index probability amplitudes
Ehrenfest's theorem	$m \frac{d}{dt} \langle \boldsymbol{r} \rangle = \langle \boldsymbol{p} \rangle$ $\frac{d}{dt} \langle \boldsymbol{p} \rangle = -\langle \nabla V \rangle$	(4.29) (4.30)	m $\boldsymbol{r}$ $\boldsymbol{p}$ V	particle mass position vector momentum potential energy

[a]Equation (4.26) uses the Dirac "bra-ket" notation for integrals involving operators. The presence of vertical bars distinguishes this use of angled brackets from that on the left-hand side of the equations. Note that $\langle a \rangle$ and $\langle \hat{a} \rangle$ are taken as equivalent.

Dirac notation

Matrix element[a]	$a_{nm} = \int \psi_n^* \hat{a} \psi_m \, \mathrm{d}x$ (4.31) $= \langle n \vert \hat{a} \vert m \rangle$ (4.32)	n,m a_{nm} ψ_n $\hat{a}$ x	eigenvector indices matrix element basis states operator spatial coordinate
Bra vector	bra state vector $= \langle n \vert$ (4.33)	$\langle \cdot \vert$	bra
Ket vector	ket state vector $= \vert m \rangle$ (4.34)	$\vert \cdot \rangle$	ket
Scalar product	$\langle n \vert m \rangle = \int \psi_n^* \psi_m \, \mathrm{d}x$ (4.35)		
Expectation	if $\Psi = \sum_n c_n \psi_n$ (4.36) then $\langle a \rangle = \sum_m \sum_n c_n^* c_m a_{nm}$ (4.37)	Ψ c_n	wavefunction probability amplitudes

[a]The Dirac bracket, $\langle n|\hat{a}|m\rangle$, can also be written $\langle \psi_n|\hat{a}|\psi_m\rangle$.